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LETTER TO THE EDITOR

Crossing formulae for critical percolation in an annulus

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Abstract
An exact formula is given for the probability that there exists a spanning cluster between opposite boundaries of an annulus in the scaling limit of critical percolation. The entire distribution function for the number of distinct spanning clusters is also derived. These results are found using Coulomb gas methods. Their forms are compared with the expectations of conformal field theory.

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Since Langlands *et al* [1] conjectured on the basis of numerical evidence that crossing probabilities between two non-overlapping segments of the boundary of a simply connected region should be conformally invariant, there has been intense interest in the scaling limit of two-dimensional percolation [2]. In [3], it was shown that this invariance was implicit in ideas of the conformal field theory, which in addition yielded an explicit formula. Further exact formulae were conjectured, by Watts [4] for the probability of a simultaneous left–right and up–down crossing, and by Pinson [5] for various crossing probabilities on the torus. The latter work used so-called Coulomb gas methods [6], which had been developed for more general two-dimensional critical systems, in parallel with those of conformal field theory. In [7], results were conjectured for the asymptotic behaviour of the probabilities that at least N_c distinct clusters cross either a rectangle or an annulus, using earlier conjectures of Saleur and Duplantier [8]. In [9], among other results, a prediction was given for the mean number of crossing clusters in the opposite limit, when this number is large.

Meanwhile, starting from another approach, Schramm [10] conjectured that the scaling limit of percolation hulls is generated by stochastic Loewner evolution (SLE_6). From this many results follow [11], including the original crossing formula and the exponents in [7]. Finally, Smirnov [12] proved the original crossing formula for site percolation on the triangular lattice, and hence the validity of the SLE_6 approach to percolation [13].

In this letter we refine the results of [7] for the annulus, presenting results for a general value of the modulus. Consider a critical percolation problem in a non-simply connected region of the plane with the topology of an annulus. The boundaries are assumed to be suitably smooth. The interior of this region may be conformally mapped into the interior of a circular annulus $R_1 \leqslant |z| \leqslant R_2$, with modulus $\tilde q \equiv (R_1/R_2)$, or into the rectangle $(0 \leqslant x < \ell, 0 < y < L)$ with the edges at $x = 0$ and $x = \ell$ identified, and $\tilde q = \mathrm{e}^{-2\pi L/\ell}$. A crossing (or spanning) cluster is one which contains a path connecting the opposite boundaries. Let $P(N_c)$ be the probability that there are exactly N_c non-overlapping such clusters. When $N_c = 1$, it is also possible for the cluster to wrap around the x-cycle on the annulus. By convention, we *do not* count such clusters as spanning.

1. Results

The crossing probability is

$$\sum_{N_c=1}^{\infty} P(N_c) = \sqrt{3}\frac{\sum_{r\in\mathbf{Z}}\left(\tilde q^{12r^2+4r+\frac{1}{4}} - \tilde q^{12r^2+8r+\frac{5}{4}}\right)}{\sum_{r\in\mathbf{Z}}\left(\tilde q^{12r^2+2r} - \tilde q^{12r^2+10r+2}\right)}. \tag{1}$$

Furthermore, we have an explicit expression for $P(N_c)$ for $N_c \geqslant 1$:

$$P(N_c) = \frac{3^{N_c-\frac{1}{2}}}{2^{2N_c-1}}\prod_{n=1}^{\infty}(1-\tilde q^{2n})^{-1}\sum_{s=0}^{\infty} A_s(N_c)\tilde q^{\frac{(N_c+s)^2}{3}-\frac{1}{12}} \tag{2}$$

where

$$A_s(N_c) = (-1)^s \sum_{r=s}^{N_c+s}\binom{r}{s}\binom{2N_c+2s}{2r}. \tag{3}$$

These results may be transformed into other expressions using various theta-function identities. For example, in terms of the conjugate modulus $q \equiv \mathrm{e}^{-\pi\ell/L}$, we find

$$\sum_{N_c=1}^{\infty} P(N_c) = \frac{\sum_{r\in\mathbf{Z}}\left(q^{6r^2+r} + q^{6r^2+5r+1} - 2q^{6r^2+3r+\frac{1}{3}}\right)}{\sum_{r\in\mathbf{Z}}\left(q^{6r^2+r} - q^{6r^2+5r+1}\right)}. \tag{4}$$

Note that for $L/\ell > \frac{1}{2}$ only a few terms need be kept in (1) and (2) for great accuracy, while for $L/\ell < \frac{1}{2}$ the same is true of (4).

Both the numerator and denominator of (1) and (4) are specializations of Jacobi theta functions, and hence may be written as infinite products. In terms of Dedekind's eta function $\eta(\tau) \equiv \tilde q^{1/24}\prod_{n=1}^{\infty}(1-\tilde q^n)$, with $\tilde q \equiv \mathrm{e}^{2\pi\mathrm{i}\tau}$, we find

$$\sum_{N_c=1}^{\infty} P(N_c) = \sqrt{3}\frac{\eta(\tau)\eta(6\tau)^2}{\eta(3\tau)\eta(2\tau)^2} = \frac{\eta(-1/\tau)\eta(-1/6\tau)^2}{\eta(-1/3\tau)\eta(-1/2\tau)^2}. \tag{5}$$

2. Coulomb gas method

Although we shall later argue that these results are indeed conformally invariant, it is simpler to first set the problem up in the periodic rectangle defined above. Consider a portion of a regular triangular lattice covering the rectangle, oriented as shown in figure 1, so that, if the lattice spacing is a, there are $2(\ell/a)+1$ columns and $(2/\sqrt{3})(L/a)+1$ rows of the lattice. Periodic boundary conditions are imposed in the x-direction, so that the rightmost column is identified with the leftmost one. Consider a critical site percolation problem on this lattice, in which sites are independently coloured red or blue with equal probability. A (blue) cluster is a set of blue sites in which every site is connected to every other by a path which traverses only blue sites. A spanning cluster is one which contains at least one site on the edge $y = 0$ and one site on the edge $y = L$. For a particular assignment of colours, let N_c be the number of

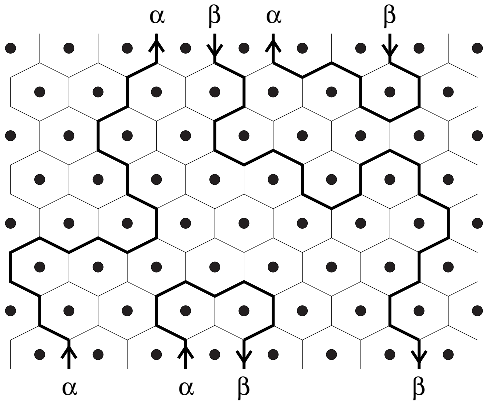

Figure 1. A triangular lattice with $\ell = 8a$, $L = 7(\sqrt{3}/2)a$. The leftmost and rightmost columns are to be identified, so the lattice has the topology of an annulus. Typical oriented spanning and non-spanning open hulls are shown, together with their boundary weights.

distinct non-overlapping spanning clusters. We are interested in the distribution $P(N_c)$ of the random variable N_c in the continuum limit $a \to 0$ for fixed L and ℓ. Scale invariance implies that it should depend only on ℓ/L.

Instead of considering the clusters, we may equivalently consider the configuration of hulls which separate them. A hull is a path on the dual lattice (in this case a honeycomb lattice) which separates blue sites from red sites. In our case, hulls can either form closed paths or be open, each end terminating at an edge. Open hulls which have ends terminating on different edges are called spanning hulls. The number of such spanning hulls is $n_c = 2N_c$. An allowable configuration of hulls is one in which each dual site is connected to either 0 or 2 neighbouring dual sites, except for the edge sites, which may be connected to either 0 or 1 neighbouring site. In addition, the number of spanning hulls must be even. The correct weights are achieved by weighting all allowable hull configurations equally.

A related model is the O(1), or Ising, model on the dual lattice, at zero temperature. In this model Ising spins $s(\mathbf{r}) = \pm 1$ reside at each site $\mathbf{r}$ of the honeycomb lattice, and the partition function is

$$Z_{O(1)} = \prod_{\mathbf{r}} \sum_{s(\mathbf{r})=\pm 1} \prod_{(\mathbf{r},\mathbf{r}')} \frac{1}{2}(1 + ts(\mathbf{r})s(\mathbf{r}')) \tag{6}$$

where the latter product is over all nearest neighbour pairs $(\mathbf{r}, \mathbf{r}')$. The 'high-temperature' expansion in powers of t, afterwards setting $t = 1$, reproduces exactly the hull configurations of the percolation model in the case when the number of spanning hulls, denoted by n_c, is even and equal to $2N_c$, but in the O(1) model n_c may also be odd. Evidently when $t = 1$, $Z_{O(1)} = 2$. Denoting by $p(n_c)$ the probability that there are exactly n_c spanning hulls, we have $P(N_c) = 2p(2n_c)$. The first factor of 2 arises because to each allowable configuration of hulls there correspond two assignments of colours. We can construct the generating function by weighting each spanning hull by a factor u. Denoting the corresponding partition function by $Z(u)$, we therefore have

$$\sum_{n_c=0}^{\infty} p(n_c) u^{n_c} = \frac{1}{2} Z(u) \tag{7}$$

$$\sum_{N_c=0}^{\infty} P(N_c)u^{2N_c} = \frac{1}{2}(Z(u) + Z(-u)). \tag{8}$$

Each hull may be assigned a random orientation, so that to each configuration of H hulls correspond 2^H configurations of oriented hulls. The weights for each orientation should be chosen so that they sum to unity (resp. u) for each (spanning) hull. For closed hulls, it is conventional [6] to assign a 'weight' $e^{\pm i\chi}$ to each dual site at which an oriented hull turns through an angle of $\pm\pi/3$, where $\chi = \pi/18$ is chosen so that the total weight for a closed hull, on summing over its orientations, is $e^{6i\chi} + e^{-6i\chi} = 1$. However, this does not correctly account for closed hulls which wrap around the x-cycle of the annulus, which would have weight $1 + 1 = 2$ according to this scheme. Such configurations can only occur when $n_c = 0$. Thus, for the time being, we assume that $n_c \geqslant 1$. The case $n_c = 0$ may be inferred afterwards using the overall normalization of the partition function.

For oriented hulls which terminate at an edge, let us assign the same weights as above for internal turnings, and in addition weights α or β to their extreme segments, as shown in figure 1. By choosing

$$\alpha = \left(\frac{\cos 6\chi}{\cos 3\chi}\right)^{1/2} e^{3i\chi'/2} \qquad \text{and} \qquad \beta = \left(\frac{\cos 6\chi}{\cos 3\chi}\right)^{1/2} e^{-3i\chi'/2} \tag{9}$$

hulls which begin and end on the same edge are counted with a weight $(\cos 6\chi/\cos 3\chi)(e^{3i\chi} + e^{-3i\chi}) = 1$, as required, while spanning hulls carry a weight $(\cos 6\chi/\cos 3\chi)(e^{3i\chi'} + e^{-3i\chi'})$, so that we should identify

$$u \equiv \cos 3\chi'/\cos 3\chi. \tag{10}$$

The factor $(\cos 6\chi/\cos 3\chi)^{1/2}$ coming from (9) is raised to a power E which is the total number of ends of open hulls, whether they be spanning or not. An open end occurs every time the neighbouring sites of the triangular lattice are of opposite colours. Since these are independently distributed, E is a sum of $O(2\ell/a)$ independent[1] random variables, and each taking the values 0 or 1 with equal probability. In the continuum limit $a/\ell \to 0$, therefore, $(\cos 6\chi/\cos 3\chi)^{E/2} \sim (\cos 6\chi/\cos 3\chi)^{\ell/2a}$, with probability 1. These contribute to the non-universal edge free energy, but not to the universal dependence on ℓ/L.

Let $Z(3\chi, 3\chi')$ be the partition function of the loop gas with the above phase factors but ignoring the factors of $(\cos 6\chi/\cos 3\chi)^{1/2}$. Then

$$\sum_{n_c=1}^{\infty} p(n_c)u^{n_c} = C_1(Z(\pi/6, \chi') - Z(\pi/6, \pi/2)) \tag{11}$$

where C_1 is a non-universal number and the second term, with $\cos 3\chi' = 0$, subtracts out the contribution with $n_c = 0$ which is incorrectly counted by the above scheme.

The configurations of the oriented loop gas are in 1-1 correspondence with those of a height model on the original triangular lattice. These heights $h(r)$ are conventionally chosen to be in $\pi\mathbf{Z}$, and are defined by the conditions that $h = 0$ at some fixed site, say (0, 0), and that it increases (decreases) by $\pm\pi$ each time an oriented hull segment is crossed. On the annulus, however, we must also allow for possible jumps $\Delta h > |\pi|$ in h across some path which spans the annulus, say along $x = -\frac{1}{4}a$. The factors $e^{\pm 3i\chi'/2}$ then accumulate to $e^{3i\chi'\Delta h/2\pi}$ on each edge.

So far, everything is finite and exact. In the conventional Coulomb gas method [6], one now assumes that in the continuum limit $(a/\ell, a/L) \to 0$ we may replace $h(r)$ by a real-valued field, with a Gaussian measure $\propto \exp(-(g/4\pi)\int(\partial h)^2\, dx\, dy)$. For the models we are considering, g is fixed to be $1 - (6\chi/\pi) = \frac{2}{3}$. We shall assume that the same is true on the

[1] Almost independent, since the sum along each edge must be even.

annulus, except that we must allow for a possible discontinuity around the x-cycle. Thus we write

$$h(x, y) = (p\pi/\ell)x + \tilde{h}(x, y) \tag{12}$$

where $p \in \mathbf{Z}$ and $\tilde{h}(x+\ell, y) = \tilde{h}(x, y)$. Substituting in this decomposition,

$$Z(3\chi, 3\chi') = C_2 \mathcal{Z}(\ell/L) \sum_{p\in\mathbf{Z}} \mathrm{e}^{3\,\mathrm{i}\chi' p}\, \mathrm{e}^{-(g/4\pi)(p\pi/\ell)^2 \ell L} \tag{13}$$

where $\mathcal{Z} \propto \int \mathcal{D}\tilde{h}\, \mathrm{e}^{-(g/4\pi)\int(\partial\tilde{h})^2\,\mathrm{d}x\,\mathrm{d}y}$ is the universal part of the partition function of a free field on the annulus, with Neumann boundary conditions, and with the constraint that $\tilde{h}(\mathbf{0}) = 0$, which removes the zero mode. The factor C_2 is non-universal, and reflects the contribution of the short-distance degrees of freedom which are integrated out in the coarse-graining assumed in adopting the Gaussian measure. It is expected to depend exponentially on the total area $(\ell L/a^2)$ and the perimeter $(2\ell/a)$, but is not expected to have non-trivial dependence on the modulus ℓ/L.

The $c = 1$ partition function $\mathcal{Z}$ is well known [14]. Writing it as $\mathrm{Tr}\,\mathrm{e}^{-\ell \hat{H}_L}$, where $\hat{H}_L$ is the quantum Hamiltonian for a free field on circle of perimeter L, it is $\prod_{n=1}^{\infty}\sum_{N=0}^{\infty} \mathrm{e}^{-\ell E_{n,N}}$ where $E_{n,N} = \left(N+\frac{1}{2}\right)(n\pi/L)$. The leading term as $\ell/L \to \infty$ comes from $N = 0$ and is proportional to $\prod_{n=1}^{\infty} \mathrm{e}^{-(\pi\ell/2L)n}$. However, this must be regularized. Apart from a cut-off dependent term which can be absorbed into C_2, it gives $\mathrm{e}^{-(\pi\ell/2L)\zeta(-1)} = q^{-\frac{1}{24}}$ where $q \equiv \mathrm{e}^{-\pi\ell/L}$. The terms with $N \geqslant 1$ give $\prod_{n=1}^{\infty}(1-q^n)^{-1}$. The zero-mode $\tilde{h}$ = constant is suppressed in the functional integral over $\tilde{h}$ since we set $\tilde{h}(\mathbf{0}) = 0$. However going from this constraint to one on the $n = 0$ mode introduces a Jacobian proportional to $(L/\ell)^{1/2}$ [14]. Finally we have

$$\mathcal{Z} = C_3 (L/\ell)^{1/2} q^{-\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)^{-1}. \tag{14}$$

Equation (13) may now be transformed using the Poisson sum formula:

$$Z(3\chi, 3\chi') = C_4 \mathcal{Z} \sum_{r\in\mathbf{Z}} \int_{-\infty}^{\infty} \mathrm{d}p\, \mathrm{e}^{2\pi\mathrm{i}pr}\, \mathrm{e}^{3\,\mathrm{i}\chi' p}\, \mathrm{e}^{-(\pi g/4)(L/\ell)p^2} \tag{15}$$

$$= C_5 q^{-\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)^{-1} \sum_{r\in\mathbf{Z}} \mathrm{e}^{-((3\chi'+2\pi r)^2)/\pi g)(\ell/L)}. \tag{16}$$

Note that the $(L/\ell)^{1/2}$ factors cancel.

Setting now $g = \frac{2}{3}$ and subtracting the contributions with $3\chi' = \frac{\pi}{6}$ and $3\chi' = \frac{\pi}{2}$, we arrive, after some algebra, at the result for the O(1) model

$$\sum_{n_c=1}^{\infty} p(n_c) = C_5 \frac{\sum_{r\in\mathbf{Z}}\left(q^{6r^2+r} - q^{6r^2+3r+\frac{1}{3}}\right)}{\prod_{n=1}^{\infty}(1-q^n)}. \tag{17}$$

Because our height model phase assignments incorrectly count loops which wrap around the x-cycle, we cannot directly compute the contribution with $n_c = 0$ and, therefore, cannot fix C_5 by demanding that $\sum_{n_c=0}^{\infty} p(n_c) = 1$. However, since $\sum_{N_c=0}^{\infty} P(N_c) = 2\sum_{N_c} p(2N_c) = 1$, it follows that $\sum_{n=0}^{\infty} p(2n+1) = \frac{1}{2}$, and we can compute this in terms of $Z(u) - Z(-u) \propto Z(\pi/6, \pi/6) - Z(\pi/6, 5\pi/6)$. This gives

$$\sum_{n_c\ \mathrm{odd}} p(n_c) = \frac{1}{2} C_5 \frac{\sum_{r\in\mathbf{Z}}\left(q^{6r^2+r} - q^{6r^2+5r+1}\right)}{\prod_{n=1}^{\infty}(1-q^n)} = \frac{1}{2} C_5 \tag{18}$$

where the last equality follows from Euler's pentagonal number theorem [15]. We conclude that $C_5 = 1$. (17) is then our main result for the probability of a crossing in the O(1) model.

For percolation, we need to compute $2\sum_{N_c=1}^{\infty} p(2N_c) \propto Z(\pi/6, \pi/6) + Z(\pi/6, 5\pi/6) - 2Z(\pi/6, \pi/2)$. This gives the main result (4).

With the knowledge that $C_5 = 1$, we may now transform these results back into series in $\tilde{q} \equiv \mathrm{e}^{-2\pi L/\ell} = \mathrm{e}^{2\pi \mathrm{i}\tau}$. Using the identity $\eta(\tau) = (-\mathrm{i}\tau)^{-\frac{1}{2}}\eta(-1/\tau)$, and the Poisson sum formula, we find

$$Z(\pi/6, \chi') = \frac{1}{2\sqrt{3}} \prod_{n=1}^{\infty} (1 - \tilde{q}^{2n})^{-1} \sum_{p \in \mathbf{Z}} \mathrm{e}^{3\,\mathrm{i}\chi' p} \tilde{q}^{(p^2-1)/12}. \tag{19}$$

Thus for the crossing probability in percolation we have

$$\sum_{N_c=1}^{\infty} P(N_c) = \frac{1}{2\sqrt{3}} \prod_{n=1}^{\infty} (1 - \tilde{q}^{2n})^{-1} \sum_{p \in \mathbf{Z}} \left(\cos\frac{\pi p}{6} + \cos\frac{5\pi p}{6} - 2\cos\frac{\pi p}{2} \right) \tilde{q}^{(p^2-1)/12}. \tag{20}$$

The expression in parentheses takes the value 3 if $p = \pm 2 \pmod{12}$, the value -3 if $p = \pm 4 \pmod{12}$, and vanishes otherwise. This leads to the first form (1) of our main result. The numerator in this expression may also be written as [16]

$$\sum_{n \in \mathbf{Z}} (-1)^n \tilde{q}^{3n^2+2n+\frac{1}{4}} = \tilde{q}^{\frac{1}{4}} \vartheta_4(2\tau \mid 6\tau) \tag{21}$$

$$= \tilde{q}^{\frac{1}{4}} \prod_{n=1}^{\infty} ((1 - \tilde{q}^{6n})(1 - \tilde{q}^{6n-1})(1 - \tilde{q}^{6n-5})) \tag{22}$$

which, after a few more manipulations, gives (5).

In order to find an explicit formula for $p(n_c)$, we should solve (10) for $\mathrm{e}^{3\,\mathrm{i}\chi'}$ in terms of u, which gives

$$\mathrm{e}^{3\,\mathrm{i}\chi'} = \mathrm{e}^{\mathrm{i}\pi/2}((1 - (\sqrt{3}u/2)^2)^{1/2} - \mathrm{e}^{-\,\mathrm{i}\pi/2}(\sqrt{3}u/2)). \tag{23}$$

Substituting this into (19), expanding in powers of u, and identifying the coefficient of u^{2N_c}, then leads to the result in (2) and (3).

A further check on our results comes from differentiating (11) with respect to u at $u = 0$ to obtain the mean number of crossing clusters. In the limit $\ell \gg L$ we find $E[N_c] \sim (\sqrt{3}/4)(\ell/L)$, in agreement with [9], and with a rigorous result of Smirnov [12] for the triangular lattice.

If instead of the periodic rectangle we have a more general annular region, in order that spanning and non-spanning hulls be counted with their correct weights α_i and β_i in (9) must be modified by factors $\mathrm{e}^{\pm\,\mathrm{i}\theta/6}$, where θ is the (signed) angle which the tangent vector at the boundary makes with the x-axis. However, since the boundaries form simple closed curves, these extra factors accumulate to unity on each edge. The calculation then proceeds as before, yielding a conformally invariant result[2].

3. Relation with conformal field theory

The crossing probability $\sum_{n_c=1}^{\infty} p(n_c)$ in the O(1) model may be expressed as a difference $Z_{++} - Z_{+-}$ of partition functions in the $n \to 1$ limit of the O(n) model, where Z_{+-} denotes

[2] Under a scale transformation $\mathbf{r} \to \lambda\mathbf{r}$, a partition function Z behaves in general as $\lambda^{c\chi/6}$, where c is the conformal anomaly number, and, for a smooth boundary, χ is the Euler number [17]. The latter vanishes for the annulus. If there are points on the boundary where it is not differentiable, however, there may be additional contributions [17]. In our case, these cancel between the $c = 1$ partition function $\mathcal{Z}$ and the Coulomb energy of the charges which accumulate at these singularities. This cancellation is connected with the fact that the overall conformal field theory has $c = 0$.

the partition function with the spins fixed in different directions on opposite edges, and Z_{++} with them fixed in the same direction. Evidently $Z_{++} = 1$, so that, from (17)

$$Z_{+-} = \frac{\sum_{r\in\mathbf{Z}}\left(q^{6r^2+3r+\frac{1}{3}} - q^{6r^2+5r+1}\right)}{\prod_{n=1}^{\infty}(1-q^n)}. \tag{24}$$

According to the general boundary conformal field theory (BCFT) [18], any partition function like this should be expressible in the form $\sum_h d_h q^h$, where h runs over all boundary scaling dimensions and d_h is a degeneracy factor. For unitary conformal theories this must be a non-negative integer, but this need not be true here. From (24) we identify the smallest value of h to be $\frac{1}{3}$: this is identified in BCFT [18] as the scaling dimension of the 'boundary condition changing operator' $\phi_{+|-}$. This is consistent with the analogous result for percolation: see [3]. From (24) we see there is also an operator with $h = 1$. This we tentatively identify as introducing a hull which wraps around the x-cycle in the O(1) model, but does not touch either edge. This should carry an O(n) index c which is not equal to either + or −, otherwise it could be absorbed at the edges. There are $n - 2 = -1$ possibilities for c, which accounts for the fact that this state occurs with degeneracy (-1) in (24).

The powers $(4p^2 - 1)/12$ in (2) are the well-known bulk multi-hull dimensions for percolation [8]. In accordance with general ideas of BCFT [18], (24) may be written as a linear combination of Virasoro characters $\chi_h(q)$ of irreducible representations of highest weight h, and, equivalently, as a combination of characters $\chi_h(\tilde{q}^2)$, related by a modular transformation. This affords an interesting example of how BCFT works in a non-minimal theory. The details will be described elsewhere [19].

4. Comparison with numerical work

Shchur [20] has estimated $P(N_c)$ for $N_c = 2, 3$ and $L = \ell$. For this value of $\tilde{q} = e^{-2\pi}$ it is sufficient to keep only the first term in (2), $P(N_c) \approx 3^{N_c-\frac{1}{2}}\tilde{q}^{(4N_c^2-1)/12}$, which gives

$$P(2) = 2.02\ldots \times 10^{-3} \text{ (exact)} \qquad 2.0(4) \times 10^{-3} \text{ (measured)} \tag{25}$$

$$P(3) = 1.71\ldots \times 10^{-7} \text{ (exact)} \qquad 1.4(5) \times 10^{-7} \text{ (measured).} \tag{26}$$

Our exact predictions fall within the (admittedly rather large) error bars.

5. Summary

We have given explicit results for the probability that N_c critical percolation clusters cross an annulus. From the point of view of conformal field theory, these results are different from the original crossing formula [3] in that they involve partition functions rather than correlation functions of boundary operators. The exponents appearing in (2) have already been derived in the limit of large modulus using a radial version of SLE [11, 13], and it would be very interesting to use these methods to verify the more detailed results of the present letter.

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References

[1] Langlands R, Pouliot P and Saint-Aubin Y 1994 *Bull. Am. Math. Soc.* **30** 1

[2] Aizenman M 1996 *STATPHYS 19 (Xiamen, 1995)* (River Edge, NJ: World Scientific)

[3] Cardy J L 1992 *J. Phys. A: Math. Gen.* **25** L201

[4] Watts G M T 1996 *J. Phys. A: Math. Gen.* **29** L363

[5] Pinson H T 1994 *J. Stat. Phys.* **75** 1167

[6] Nienhuis B 1987 *Phase Transitions and Critical Phenomena* vol 11, ed C Domb and J L Lebowitz (New York: Academic)

[7] Cardy J L 1998 *J. Phys. A: Math. Gen.* **31** L105

[8] Saleur H and Duplantier B 1987 *Phys. Rev. Lett.* **38** 2325

[9] Cardy J L 2000 *Phys. Rev. Lett.* **84** 3507

[10] Schramm O 2000 *Israel J. Math.* **118** 221

[11] Lawler G F, Schramm O and Werner W 2001 *Acta Math.* **187** 237
Lawler G F, Schramm O and Werner W 2001 *Acta Math.* **187** 275
Lawler G F, Schramm O and Werner W 2002 *Ann. Inst. H. Poincaré* PR **38** 109
Lawler G F, Schramm O and Werner W 2001 *Preprint* arXiv:math.PR/0108211

[12] Smirnov S 2001 *C. R. Acad. Sci. Paris Sr. I Math.* **333** 239

[13] Smirnov S and Werner W 2001 *Math. Res. Lett.* **8** 729

[14] Itzykson C and Drouffe J-M 1989 *Statistical Field Theory* vol 2 (Cambridge: Cambridge University Press)

[15] See, for example, McKean H and Moll V 1999 *Elliptic Curves* (Cambridge: Cambridge University Press) p 143

[16] McKean H and Moll V 1999 *Elliptic Curves* (Cambridge: Cambridge University Press) p 125, 135

[17] Cardy J L and Peschel I 1988 *Nucl. Phys.* B **300** 377

[18] Cardy J L 1989 *Nucl. Phys.* B **324** 581
Cardy J L 1989 *Adv. Stud. Pure Math.* **19** 127

[19] Cardy J L in preparation

[20] Shchur L N and Kosyakov S S 1998 *Nucl. Phys.* B (Proc. Suppl.) **63A–C** 664
Shchur L N 1999 *Preprint* arXiv:cond-mat/9906013